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STRESS AND CONCENTRATED FORCES IN THIN ANNULAR PLATES†

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Exact solutions of some three-dimensional problem of the deformation of a thin annular plate fixed along the outer contour and loaded along the inner contour in a certain way are obtained using functions of a complex variable. The stress distributions on it under various concentrated forces are obtained by taking the limit when the radius of the inner circumference tends to zero.

Concentrated forces in thin plates should be distinguished by their effect on deformable bodies, taking into account their origin and technical, technological or scientific applications. One of the types of concentrated forces is linked to the displacement of a rigid disc – soldered into a thin plate – used to fastening instruments or any other equipment on the plate, or a rigid rod used as a fixing element or axle. Another type of concentrated force arises during the instantaneous burn through of a thin cylindrical channel in a plate. Finally, the third type of concentrated force arises when there is an increase in the diameter of the hole in the plate due to an increase in the diameter of a rivet when it is deformed or when acted upon by special tools in technological operations.

The plate is usually assumed to be of infinite length when analysing the action of concentrated forces in the mechanics of solids. This assumption is not always correct. For instance, when investigating concentrated forces in the first of the types considered it leads to a trivial solution, even though it is obvious that the stresses (and strains) in such a case will by no means be zero for a plate of finite size and fixed along the outer contour. Furthermore, the *a priori* specification of the stress distribution along the contour of the aperture, subjected to concentrated forces, can in some cases appear not to correspond to reality.

1. REPRESENTATION OF THE COMPONENTS OF THE DISPLACEMENT VECTOR AND STRESS TENSOR IN PROBLEMS OF DEFORMATION OF THIN PLATES WHEN USING THREE COMPLEX POTENTIALS

The theory of functions of a complex variable (TFCV) is widely used in the theory of elasticity, primarily for solving plane problems [1]. To further develop the use of TFCV methods in the theory of elasticity a third complex potential was introduced in addition to the two Kolosov–Muskhelishvili complex potentials, and formulations of three-dimensional problems of elasticity for thin plates of variable thickness, taking into account the above conditions, were investigated [2]. New solutions for classical problems of the uniaxial tension of a thin plate with a free circular hole and free elliptic one in a three-dimensional formulation were obtained by using three complex potentials [3, 4].

We will introduce the main relations, linking the components of the displacement vector (u_1, u_2, u_3) in a rectangular system of coordinates $OX_1X_2X_3$ (or $(u_\rho, u_\vartheta, u_3)$ in a system of cylindrical polar coordinates (ρ, ϑ, x_3)) and a stress tensor $(\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{33})$ in a rectangular system of coordinates $((\sigma_{\rho\rho}, \sigma_{\vartheta\vartheta}, \sigma_{\rho\vartheta}, \sigma_{\rho 3}, \sigma_{\vartheta 3}, \sigma_{33})$ in cylindrical polar coordinates) with three complex potentials, for the further use of TFCV methods to investigate the stress–strain state.

The quantities method, expressed in terms of the three complex potentials $\varphi(z)$, $\phi(z)$, $f(z)$, have the form [2]

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$$4\mu(u_1 + iu_2) = f(z) - 2[\varphi(z) + z\overline{\varphi'(z)} + \overline{\phi(z)}] \tag{1.1}$$

$$\sigma_{11} + \sigma_{22} + \sigma_{33} = 2(1+\nu)[\varphi'(z) + \overline{\varphi'(z)}] \tag{1.2}$$

$$\sigma_{11} + \sigma_{22} = \frac{1}{2}[f'(z) + \overline{f'(z)}] - 2(1-2\nu)[\varphi'(z) + \overline{\varphi'(z)}] \tag{1.3}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\varphi''(z) + \phi'(z)] \tag{1.4}$$

$$\sigma_{13} - i\sigma_{23} = 2x_3\left[2(1-\nu)\varphi''(z) - \frac{1}{4}f''(z)\right] \tag{1.5}$$

$$\sigma_{33} = 2(2-\nu)[\varphi'(z) + \overline{\varphi'(z)}] - \frac{1}{2}[f'(z) + \overline{f'(z)}] \tag{1.6}$$

in a rectangular Cartesian system of coordinates, where $z = x_1 + ix_2$ and ν is Poisson's ratio. Complex conjugate quantities are denoted by a bar.

We will also introduce an expression for the component of the strain tensor ε_{33}

$$\varepsilon_{33} = \frac{2(1+\nu)}{E}\left\{2(1-\nu)[\varphi'(z) + \overline{\varphi'(z)}] - \frac{1}{4}[f'(z) + \overline{f'(z)}]\right\} \tag{1.7}$$

where E is Young's modulus. Moreover, the displacement in the third direction is given by the expression

$$u_3 = \varepsilon_{33}x_3 \tag{1.8}$$

The relations between the components of the displacement vector and the components of the stress tensor in the systems of rectangular and cylindrical polar coordinates have the following form

$$u_\rho + iu_\vartheta = (u_1 + iu_2)e^{-i\vartheta} \tag{1.9}$$

$$\sigma_{\rho\rho} + \sigma_{\vartheta\vartheta} + \sigma_{33} = \sigma_{11} + \sigma_{22} + \sigma_{33} \tag{1.10}$$

$$\sigma_{\vartheta\vartheta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\vartheta} = (\sigma_{22} - \sigma_{11} + 2i\sigma_{12})e^{2i\vartheta} \tag{1.11}$$

$$\sigma_{\rho 3} - i\sigma_{\vartheta 3} = (\sigma_{13} - i\sigma_{23})e^{i\vartheta} \tag{1.12}$$

In particular, it follows from Eq. (1.10) that

$$\sigma_{\rho\rho} + \sigma_{\vartheta\vartheta} = \sigma_{11} + \sigma_{22} \tag{1.13}$$

The complex potentials can be given in the form [2]

$$\varphi(z) = Az\ln z + \alpha\ln z + \sum_{k=-\infty}^{+\infty} a_k z^k \tag{1.14}$$

$$\phi(z) = \beta\ln z + \sum_{k=-\infty}^{+\infty} b_k z^k \tag{1.15}$$

$$f(z) = \gamma\ln z + \sum_{k=-\infty}^{+\infty} c_k z^k \tag{1.16}$$

where A is a real quantity.

2. STRESSES IN AN ANNULAR PLATE, FIXED ALONG THE OUTER CONTOUR WHEN A CENTRAL RIGID WASHER SOLDERED INTO THE PLATE IS DISPLACED

Consider an annular plate with an inner radius r and an outer radius R. Let us assume that, soldered into the inside of the annular plate is a rigid disc of radius r and that the plate is fixed along the outer contour. Further, we will also assume that the rigid inclusion is shifted in a certain direction.

We will place the centre of the plate at the origin of the rectangular Cartesian and cylindrical polar systems of coordinates. The boundary conditions for the problem can be reduced to two relations

$$(u_1 + iu_2)|_{\rho = r} = U_0 + iV_0, \quad (u_1 + iu_2)|_{\rho = R} = 0 \tag{2.1}$$

We will assume that the components of the displacement vector along the contour of the inner hole of the plate are constant quantities.

It is easy to see that the rigid fixing of the central absolutely rigid inclusion leads to the boundary condition

$$u_3|_{\rho = r} = 0 \tag{2.2}$$

We will also assume that the condition

$$\sigma_{33}|_{\rho = R} = 0 \tag{2.3}$$

is satisfied on the outer contour.

We will substitute Eqs (1.14)–(1.16) into Eq. (1.1) to implement boundary conditions (2.1). We obtain

$$\begin{aligned}4\mu(u_1 + iu_2) &= \gamma(\ln\rho + i\vartheta) + \sum_{k=-\infty}^{+\infty} c_k\rho^k e^{ik\vartheta} - \\ &-2\left[A\rho e^{i\vartheta}(\ln\rho + i\vartheta) + \alpha(\ln\rho + i\vartheta) + \sum_{k=-\infty}^{+\infty} a_k\rho^k e^{ik\vartheta}\right] - \\ &-2\rho e^{i\vartheta}\left[A(\ln\rho - i\vartheta) + \frac{\bar{\alpha}}{\rho} + \sum_{k=0}^{+\infty} k\bar{a}_k\rho^{k-1}e^{i(k-1)\vartheta} - \sum_{k=-1}^{-\infty} k\bar{a}_k\rho^{k-1}\right] - \\ &-2\left[\bar{\beta}(\ln\rho - i\vartheta) + \sum_{k=0}^{+\infty} \bar{b}_k\rho^k e^{-ik\vartheta} + \sum_{k=-1}^{-\infty} \bar{b}_k\rho^k e^{-ik\vartheta}\right]\end{aligned} \tag{2.4}$$

The sum of coefficients of $i\vartheta$ must vanish [1, 2] by virtue of the uniqueness of the displacements

$$\gamma - 2\alpha + 2\bar{\beta} = 0 \tag{2.5}$$

Collecting the coefficients of e^0 and taking into account conditions (2.1) we have

$$\begin{aligned}&(\gamma - 2\alpha - 2\bar{\beta})\ln r + A_0 - 4\bar{a}_2 r^2 = 4\mu(U_0 + iV_0)\\ &(\gamma - 2\alpha - 2\bar{\beta})\ln R + A_0 - 4\bar{a}_2 R^2 = 0\end{aligned} \tag{2.6}$$

where $A_0 = c_0 - 2a_0 - 2\bar{b}_0$).

The sums of the coefficients of $e^{i\vartheta}$ in Eq. (2.4) lead to two equations, which follow from boundary conditions (2.1)

$$\begin{aligned}&c_1 r - 2a_1 r - 4Ar\ln r - 2Ar - 2\bar{a}_1 r - 2\frac{\bar{b}_{-1}}{r} = 0\\ &c_1 R - 2a_1 R - 4AR\ln R - 2AR - 2\bar{a}_1 R - 2\frac{\bar{b}_{-1}}{R} = 0\end{aligned} \tag{2.7}$$

and for $e^{2i\vartheta}$ they lead to the equations

$$
\begin{aligned}
&c_2 r^2 - 2a_2 r^2 - 2\bar{\alpha} - 2\frac{\bar{b}_{-2}}{r^2} = 0 \\
&c_2 R^2 - 2a_2 R^2 - 2\bar{\alpha} - 2\frac{\bar{b}_{-2}}{R^2} = 0
\end{aligned}
\tag{2.8}
$$

System (2.5)–(2.8), consisting of seven equations, contains eleven unknown coefficients

$$\alpha, \beta, \gamma, A_0, a_2, b_{-2}, c_2, A, a_1, b_{-1}, c_1 \tag{2.9}$$

We draw attention to the fact that none of the coefficients (2.9) appear in the combination of coefficients for other values of k in $e^{ik\vartheta}$

$$
\begin{aligned}
&c_k \rho^k - 2a_k \rho^k - 2\frac{\bar{a}_{-(k-2)}}{\rho^{k-2}} - 2\frac{\bar{b}_{-k}}{\rho^k} = 0, \quad k \geq 3 \\
&\frac{c_{-k}}{\rho^k} - 2\frac{a_{-k}}{\rho^k} - 2(k+2)\bar{a}_{k+2}\rho^{k+1} - 2\bar{b}_k \rho^k = 0, \quad k \geq 1
\end{aligned}
\tag{2.10}
$$

We will supplement the above relations with equations derived from conditions (2.2) (taking into account Eqs (1.7), (1.8) and (2.3)). Thus, we substitute Eqs (1.14)–(1.16) into the above boundary conditions. It is obvious that the equations containing unknowns from the set (2.9) are of the main interest. These equations are

$$8(1-\nu)[2A(\ln r + 1) + (a_1 + \bar{a}_1)] - (c_1 + \bar{c}_1) = 0 \tag{2.11}$$

$$8(1-\nu)\left(2a_2 r + \frac{\bar{\alpha}}{r}\right) - \left(2c_2 r + \frac{\bar{\gamma}}{r}\right) = 0 \tag{2.12}$$

$$4(2-\nu)[2A(\ln R + 1) + (a_1 + \bar{a}_1)] - (c_1 + \bar{c}_1) = 0 \tag{2.13}$$

$$4(2-\nu)\left(2a_2 R + \frac{\bar{\alpha}}{R}\right) - \left(2c_2 R + \frac{\bar{\gamma}}{R}\right) = 0 \tag{2.14}$$

We note that the other equations obtained from boundary conditions (2.2) and (2.3) do not contain unknowns from the set (2.9) and are not mentioned here for the following reason. The boundary conditions are homogeneous, and hence the equations, not presented here, for the unknowns, not contained in the set (2.9), together with Eqs (2.1) can only form a homogeneous system. The determinant of such a system will not vanish owing to the arbitrariness of the quantities governed by it. This will result in a trivial solution, i.e. all unknown coefficients of the expansion that are not part of the set (2.9) are equal to zero.

The system of equations (2.25)–(2.8), (2.11)–(2.14) consists of two sub-systems: one contains Eqs (2.5), (2.6), (2.8), (2.12) and (2.14) with unknowns

$$\alpha, \beta, \gamma, A_0, a_2, b_{-2}, c_2 \tag{2.15}$$

and the second consists of Eqs (2.7), (2.11) and (2.13) for the set of unknowns

$$A, a_1, b_{-1}, c_1 \tag{2.16}$$

We will first consider the second system. It is easily transformed to the form

$$
\begin{aligned}
&-2r(2\ln r+1)A-2r(a_1+\bar{a}_1)-\frac{2}{r}\bar{b}_{-1}+rc_1 = 0\\
&-2R(2\ln R+1)A-2R(a_1+\bar{a}_1)-\frac{2}{R}\bar{b}_{-1}+Rc_1 = 0\\
&16(1-\nu)(\ln r+1)A+8(1-\nu)(a_1+\bar{a}_1)-(c_1+\bar{c}_1) = 0\\
&8(2-\nu)(\ln R+1)A+4(2-\nu)(a_1+\bar{a}_1)-(c_1+\bar{c}_1) = 0
\end{aligned}
\tag{2.17}
$$

We obtain two systems of equations in the real and imaginary parts of the unknown coefficients of the expansion. Their matrices have the following form

$$
\begin{Vmatrix}
-2r(2\ln r+1) & -4r & -\frac{2}{r} & r\\
-2R(2\ln R+1) & -4R & -\frac{2}{R} & R\\
16(1-\nu)(\ln r+1) & 16(1-\nu) & 0 & -2\\
8(2-\nu)(\ln R+1) & 8(2-\nu) & 0 & -2
\end{Vmatrix}
\quad \text{and} \quad
\begin{Vmatrix}
0 & 0 & \frac{2}{r} & r\\
0 & 0 & \frac{2}{R} & R\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{Vmatrix}
\tag{2.18}
$$

respectively.

It can be prove directly that the determinant of the first matrix is non-zero, and thus the real parts of the unknowns, belonging to the set (2.16), vanish. From the system of equations for their imaginary parts, it is obviously that the coefficient a_1 is determined, apart from the imaginary part of this coefficient. However, this has no influence on the stress–strain state in the annular plate, since all the quantities contain the real part of the coefficient a_1.

We will now consider the first system. We will transform this system in order to simplify the mathematics. We will first consider Eqs (2.6) taking into account Eq. (2.5). As a result we obtain

$$
\begin{aligned}
&A_0-\ln r(4\alpha-2\gamma)-4r^2\bar{a}_2 = 4\mu(U_0+iV_0)\\
&A_0-\ln R(4\alpha-2\gamma)-4R^2\bar{a}_2 = 0
\end{aligned}
\tag{2.19}
$$

By subtracting the second equation from the first we obtain

$$
(\ln R-\ln r)(4\alpha-2\gamma)+4(R^2-r^2)\bar{a}_2 = 4\mu(U_0+iV_0) \tag{2.20}
$$

We now transform Eq. (2.8) by multiplying the first equation by r^2 and the second equation by R^2. We then subtract the second equation from the first, after which the equation obtained is reduce by R^2-r^2; as a result we have

$$
2\bar{\alpha}+2(R^2+r^2)a_2-(R^2+r^2)c_2 = 0 \tag{2.21}
$$

By supplementing Eqs (2.20) and (2.21) with Eqs (2.12) and (2.14), also containing the unknowns α, γ, a_2, c_2, we obtain a system of four equations for the required unknowns.

We now divide the first equation of (2.8) by r^2 and the second equation by R^2 and subtract one equation from the other. As a result we obtain

$$
b_{-2} = -\frac{R^2r^2}{R^2+r^2}\alpha \tag{2.22}
$$

The quantity β is determined from the relation

$$
\bar{\beta} = \alpha-\gamma/2 \tag{2.23}
$$

after which we have from the second equation of (2.19)

$$
A_0 = 4\bar{\beta}\ln R+4\bar{a}_2R^2 \tag{2.24}
$$

The solution of the system (2.20), (2.21), (2.12) and (2.14) together with Eqs (2.22)–(2.24) gives the following expression for the unknowns from the set (2.15)

$$
\begin{aligned}
\alpha &= \frac{1}{\Delta}[(3-2\nu)R^4 + 2\nu R^2 r^2 - (3-4\nu)r^4](U_0 + iV_0) = A_1(U_0+iV_0) \\
a_2 &= \frac{1}{\Delta}[(R^2-r^2) - \nu(R^2+r^2)](U_0 - iV_0) = A_2(U_0 - iV_0) \\
\beta &= \frac{1}{\Delta}[(-9+18\nu-8\nu^2)(R^4-r^4) + 4\nu R^2 r^2](U_0 - iV_0) = B_1(U_0 - iV_0) \\
b_{-2} &= \frac{1}{\Delta}[-(3-2\nu)R^4 - 2\nu R^2 r^2 + (3-4\nu)r^4]\frac{R^2 r^2}{R^2+r^2}(U_0+iV_0) = B_2(U_0+iV_0) \qquad (2.25)\\
\gamma &= \frac{2}{\Delta}[(12-20\nu+8\nu^2)R^4 - 2\nu R^2 r^2 - (12-22\nu+8\nu^2)r^4](U_0+iV_0) = C_1(U_0+iV_0) \\
c_2 &= \frac{2}{\Delta}[(4-3\nu)(R^2-r^2)](U_0-iV_0) = C_2(U_0-iV_0) \\
A_0 &= \frac{4}{\Delta}\{[(-9+18\nu-8\nu^2)(R^4-r^4) + 4\nu R^2 r^2]\ln R + (R^2-r^2) - \nu(R^2+r^2)\}(U_0+iV_0)
\end{aligned}
$$

where

$$
\Delta = \frac{1}{\mu}\left\{[-(3-4\nu)(3-2\nu)(R^4-r^4) + 4\nu r^2 R^2]\ln\frac{R}{r} + (R^2-r^2)^2 - \nu(R^4-r^4)\right\}
$$

Note that the quantity A_0, which is a combination of constants, determined above (see Eqs (2.26)), is used as such a combination only to determine the displacements and is not part of any other relations; hence there is no need to determine its components.

Thus, the complex potentials in the problem take the form

$$
\begin{aligned}
\varphi(z) &= \alpha\ln z + a_0 + a_2 z^2 \\
\phi(z) &= \beta\ln z + b_0 + b_{-2}z^{-2} \qquad (2.26)\\
f(z) &= \gamma\ln z + c_0 + c_2 z^2
\end{aligned}
$$

where all the coefficients, apart from a_0, b_0, c_0, are determined by the first six equations of (2.25), and the linear combination of the coefficients is determined by the last equation of (2.25).

It is easy to obtain all the quantities completely characterizing the stress–strain state of the deformable body in the problem considered by using the representation of the complex potentials (2.26). For instance, the complex displacement $(u_1 + iu_2)$, by virtue of Eqs (1.1), (2.5), (2.23) and (2.24), can be easily found from the expression

$$
4\mu(u_1+iu_2) = 4\bar\beta\ln\frac{R}{\rho} + 4\bar a_2(R^2-\rho^2) - \left[2\frac{\bar b_{-2}}{\rho^2} + 2\bar\alpha - (c_2 - 2a_2)\rho^2\right]e^{2i\vartheta}
$$

However, the complex form of the displacement vector is inconvenient for different estimates, and we will therefore separate the complex and real parts, using the first six equations from (2.25) in this expression. As a result we have

$$
4\mu(u_1+iu_2) = \chi_1 U_0 + \chi_2(U_0\cos 2\vartheta + V_0\sin 2\vartheta) + i[\chi_1 V_0 + \chi_2(U_0\sin 2\vartheta - V_0\cos 2\vartheta)] \qquad (2.27)
$$

where

$$\chi_1 = 4B_1\ln\frac{R}{\rho} + 4A_2(R^2-\rho^2), \quad \chi_2 = -2\frac{B_2}{\rho^2} - 2A_1 + (C_2-2A_2)\rho^2$$

Using relations (1.2)–(1.6) and also (1.10)–(1.13) and (2.28)–(2.33) we will obtain the main combinations of components of the stress tensor

$$\sigma_{\rho\rho}+\sigma_{\vartheta\vartheta} = \left[\left(\frac{C_1}{\rho}+2C_2\rho\right)-4(1-2\nu)\left(\frac{A_1}{\rho}+2A_2\rho\right)\right]W_1$$

$$\sigma_{\vartheta\vartheta}-\sigma_{\rho\rho}+2i\sigma_{\rho\vartheta} = 2\left[-2\frac{B_2}{\rho^3}-\frac{A_1-B_1}{\rho}+2A_2\rho\right]W_1 - 2i\left[2\frac{B_2}{\rho^3}+\frac{A_1+B_1}{\rho}+2A_2\rho\right]W_2$$

and also the components of this tensor

$$\sigma_{\rho\rho} = \frac{1}{2}\left\{4\frac{B_2}{\rho^3}+\frac{C_1-2B_1-2(1-4\nu)A_1}{\rho}+2[C_2-2(3-4\nu)A_2]\rho\right\}W_1$$

$$\sigma_{\vartheta\vartheta} = \frac{1}{2}\left\{-4\frac{B_2}{\rho^3}+\frac{C_1+2B_1-2(3-4\nu)A_1}{\rho}+2[C_2-2(1-4\nu)A_2]\rho\right\}W_1$$

$$\sigma_{33} = \left\{\frac{4(2-\nu)A_1-C_1}{\rho}+2[4(2-\nu)A_2-C_2]\rho\right\}W_1$$

$$\sigma_{\rho\vartheta} = -\left[2\frac{B_2}{\rho^3}+\frac{A_1+B_1}{\rho}+2A_2\rho\right]W_2 \tag{2.28}$$

$$\sigma_{\rho 3} = 2x_3\left[\frac{C_1-8(1-\nu)A_1}{4\rho^2}+4(1-\nu)A_2-\frac{C_2}{2}\right]W_1$$

$$\sigma_{\vartheta 3} = -2x_3\left[\frac{C_1-8(1-\nu)A_1}{4\rho^2}-4(1-\nu)A_2+\frac{C_2}{2}\right]W_2$$

where

$$W_1 = U_0\cos\vartheta + V_0\sin\vartheta, \quad W_2 = -U_0\sin\vartheta + V_0\cos\vartheta$$

The expression for the component of the strain tensor ε_{33}, necessary to form boundary condition (2.2), has the form

$$\varepsilon_{33} = \frac{2(1+\nu)}{E}\left\{\frac{8(1-\nu)A_1-C_1}{2\rho}+[8(1-\nu)A_2-C_2]\rho\right\}W_1 \tag{2.29}$$

Equations (2.28) and (2.29) provide the complete solution of the problem. It can be established by direct proof that these equations, without exception, satisfy all the equilibrium equations, the compatibility conditions and the boundary conditions.

The stresses on the contour of the circular aperture in the plate are given by the expression obtained from (2.28) for $\rho = r$.

$$\sigma_{\rho\rho} = \frac{4}{\Delta}\left[\frac{(3-2\nu)(1-\nu)}{r}R^4+2(1-\nu)R^2r-(5-2\nu)(1-\nu)r^3\right]W_1$$

$$\sigma_{\vartheta\vartheta} = \frac{4}{\Delta}\left[\frac{(3-2\nu)\nu}{r}R^4+2\nu R^2r-(5-2\nu)r^3\right]W_1$$

$$\sigma_{33} = \frac{4\nu}{\Delta}\left[\frac{3-2\nu}{r}R^4 + 2R^2 r - (5-2\nu)r^3\right]W_1 \tag{2.30}$$

$$\sigma_{\rho\vartheta} = \frac{4}{\Delta}\left[\frac{(3-2\nu)(1-\nu)}{r}R^4 - (2-\nu)rR^2 - (1-4\nu+2\nu^2)r^3\right]W_2$$

$$\sigma_{\rho 3} = -\frac{4\nu x_3}{\Delta}[(5-4\nu)R^2 + (3-4\nu)r^2]W_1, \quad \sigma_{\vartheta 3} = 0$$

The stress distribution along the contour of the circular aperture when a concentrated force of the type considered acts on it can be derived from Eqs (2.30) by taking the limit as $r \to 0$. Taking this limit, we obtain for the non-zero components of the stress tensor

$$\sigma_{\rho\rho} = 4\mu\frac{1-\nu}{3-4\nu}\frac{1}{r\ln r}W_1$$

$$\sigma_{\vartheta\vartheta} = \sigma_{33} = 4\mu\frac{\nu}{3-4\nu}\frac{1}{r\ln r}W_1$$

$$\sigma_{\rho\vartheta} = 4\mu\frac{1-\nu}{3-4\nu}\frac{1}{r\ln r}W_2, \quad \sigma_{\rho 3} = -4\mu\frac{\nu(5-4\nu)}{(3-4\nu)(3-2\nu)}\frac{x_3}{r\ln r}W_1$$

However, if we first take the limit as $R \to \infty$ in Eqs (2.28), these quantities as a result will have the order of $1/\ln R$ and, thus, they will, as might have been expected, tend to zero in this case, which results in a trivial solution in the case of an infinite plate. Hence, the assumption that the plate is infinite in the problems considered crucial.

It follows from an analysis of the general solution (2.28) and, particularly, along the edge of the cut (Eqs (2.30)) that the shear components of the stress tensor, lined with the third coordinate, in absolute value are quantities of the same order as the others. Furthermore, in some cases, noted previously in [4], they may considerably exceed in absolute value all the other components of the stress tensor. This indicates, that the problem considered here is a three-dimensional non-axisymmetric problem of the theory of elasticity without any constraints. In this case the component of the stress tensor σ_{33} on the free surface of the plate can be non-zero, but for this the components $\sigma_{\rho 3}$, $\sigma_{\vartheta 3}$, σ_{33} must necessarily satisfy the third equilibrium equation, which also occurs in this case.

Moreover, the solution obtained can be easily transformed into the solution of the corresponding problem of the generalized state of plane stress. Poisson's ratio ν, as follows from results obtained previously [3], is considered to be equal to zero in problems of the generalized state of plane stress. It has been established that the components of the stress tensor $\sigma_{\rho 3}$, $\sigma_{\vartheta 3}$, σ_{33}, defined by Eq. (2.28), are proportional to ν. Thus, whereas according to the theory of the generalized state of plane stress along the free surface of a plate of finite thickness we assume $\sigma_{\rho 3} = 0$, $\sigma_{\vartheta 3} = 0$ (and this is only possible when $\nu = 0$), then in a traditional way we obtain from the third equation of equilibrium that everywhere $\sigma_{33} = 0$, which corresponds completely to the results obtained.

3. THE ACTION OF A UNIFORM PRESSURE ON THE CONTOUR OF AN ANNULAR PLATE FIXED ALONG THE OUTER CONTOUR

We will consider the problem of the action of normal and shear stresses on inner contour of an annular plate of constant width, fixed along the outer contour.

We will assume that the annular plate of unit thickness and outer radius R has a central circular of hole radius r. We will assume that the normal and shear stresses are given along the inner contour and that the outer contour of the plate is fixed.

We will introduce a cylindrical polar system of coordinates (ρ, ϑ, x_3). Then the boundary conditions of the problem will take the form

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta}\big|_{\rho=r} = p - iq, \quad u_\rho + iu_\vartheta\big|_{\rho=R} = 0 \tag{3.1}$$

Further, we will assume that everywhere

$$\sigma_{33} = 0 \tag{3.2}$$

These combinations of values are expressed in terms of complex potentials using relations [2]

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta} = \frac{1}{4}[f'(z) + \overline{f'(z)}] - (1-2\nu)[\varphi'(z) + \overline{\varphi'(z)}] - [\bar{z}\varphi''(z) + \phi'(z)]e^{2i\vartheta} \tag{3.3}$$

and Eqs (1.1) (taking (1.9) into account) and (1.6), respectively.

The solution of the problem will be sought using power series [1]. We will take the complex potentials in the form (1.14)–(1.16). Substituting these into boundary conditions (3.1) and (3.2) we obtain the following system of equations

$$\begin{gathered}\frac{1}{4}(c_1 + \bar{c}_1) - (1-2\nu)[2(\ln r + 1)A + (a_1 + \bar{a}_1)] - A + \frac{b_{-1}}{r^2} = p - iq \\ Rc_1 - 2R(a_1 + \bar{a}_1) - 2R(2\ln R + 1)A - 2\frac{\bar{b}_{-1}}{R} = 0 \\ 2(2-\nu)A\ln\rho = 0, \quad 2(2-\nu)(a_1 + \bar{a}_1) - \frac{1}{2}(c_1 + \bar{c}_1) = 0\end{gathered} \tag{3.4}$$

We will explain the origin of the equations in this system. The first equation of (3.4) is obtained as a consequence of the choice of the coefficients of the expansion for e^0 in the first boundary condition (3.1), and the second equation of (3.4) is obtained as a consequence of the choice of the coefficients of the expansion for R in the second boundary condition (3.1). The last two equations of (3.4) are obtained for the implementation of boundary condition (3.2) as a combination of expansion terms including the unknown coefficients of the expansion, which are part of Eq. (3.4) (see Eq. (2.13) in which R should be replaced by ρ and its terms dependent on and independent of ρ should be separately equal to zero).

We obtain directly $A = 0$ from the third equation of (3.4). We obtain the following two systems of equations, taking this into account and dividing the required coefficients by the real and imaginary parts: for the real parts of the coefficient

$$\begin{aligned} r^2c_{11} - 4(1-2\nu)r^2a_{11} + 2b_{-11} &= 2pr^2 \\ c_{11} - 4(2-\nu)a_{11} &= 0 \\ R^2c_{11} - 4R^2a_{11} - 2b_{-11} &= 0 \end{aligned}$$

and for their imaginary parts

$$\begin{aligned} b_{-12} &= -qr^2 \\ R^2c_{12} + 2b_{-12} &= 0 \end{aligned}$$

We obtain expressions for coefficients of the expansions of the complex potentials by solving these systems. As a result, the complex potentials take the form

$$\begin{aligned} \varphi(z) &= a_0 + \left(\frac{pr^2}{2d} + ia_{12}\right)z \\ \phi(z) &= b_0 + \left[\frac{(1-\nu)pR^2r^2}{d} - iqr^2\right]\frac{1}{z} \\ f(z) &= c_0 + \left[\frac{2(2-\nu)pr^2}{d} + 2iq\frac{r^2}{R^2}\right]z \end{aligned}$$

where

$$d = (1-\nu)R^2 + (1+\nu)r^2$$

Note that the imaginary part of the coefficient a_1 cannot be determined from the given system of equations. However, it is not very important, since, as we shall see below, it does not participate in the representation of the components of the displacements vector and the stress tensor.

It can be easily seen by directly substituting the expressions obtained for the complex potentials into Eq. (1.6) that Eq. (3.2) is correct everywhere.

We obtain the components of the stress tensor by substituting these expressions into Eq. (3.3)

$$\sigma_{\rho\rho} = \frac{(1+\nu)pr^2}{(1-\nu)R^2+(1+\nu)r^2} + \frac{(1-\nu)pR^2r^2}{(1-\nu)R^2+(1+\nu)r^2}\frac{1}{\rho^2}, \quad \sigma_{\rho\vartheta} = q\frac{r^2}{\rho^2} \tag{3.5}$$

Since the expression for $\sigma_{\vartheta\vartheta} + i\sigma_{\rho\vartheta}$ differs from (3.3) only in the sign of the multiplier of the exponential function [2], the expression for $\sigma_{\vartheta\vartheta}$ differs from the expression for $\sigma_{\rho\rho}$ only in having a minus sign instead of a plus sign in front of the second term in the first formula of (3.5).

It can be established by direct proof that the components of the stresses tensor obtained satisfy all the equilibrium equations and compatibility conditions and the first boundary condition of (3.1). Furthermore, it is necessary to ensure that the second boundary condition of (3.1) is satisfied, so that the complex potentials obtained determine the solution of the problem. We will consider this in detail.

Taking into account Eq. (1.9) for the components of the displacements vector instead of Eq. (1.1), we obtain an expression, which, apart from everything else, contains the linear combination $c_0 - 2a_0 - 2b_0$, which, taking into account the second boundary condition of (3.1), is defined perfectly uniquely. In this case it is equal to zero, and this enables us, by separating the real and imaginary parts in this expression, to obtain finally

$$u_\rho = \frac{1}{2\mu}\frac{(1-\nu)pr^2}{(1-\nu)R^2+(1+\nu)r^2}\left(\rho - \frac{R^2}{\rho}\right), \quad u_\vartheta = \frac{1}{2\mu}\frac{qr^2}{R^2}\left(\rho - \frac{R^2}{\rho}\right)$$

It can be seen that the principal vector of the forces acting on the inner contour vanishes, but the value of this force, acting on half the contour, for example, in the range $-\pi/2 \le \vartheta \le \pi/2$, is equal to

$$-\int_{-\pi/2}^{\pi/2} pr\cos\vartheta d\vartheta = -2pr$$

We will assume that $r \to 0$, but in addition we will assume that the limit value of the force $F = -pr \neq 0$ and remains a certain constant value. Then in the limit as $\rho = r \to 0$ we have

$$\sigma_{\rho\rho} = \frac{F}{r}, \quad \sigma_{\vartheta\vartheta} = -\frac{F}{r}, \quad u_\rho = -\frac{F}{2\mu}$$

Similarly, assuming that $Q = qr \neq 0$ in the limit as $\rho = r \to 0$ we have

$$\sigma_{\rho\vartheta} = \frac{Q}{r}, \quad u_\vartheta = -\frac{Q}{2\mu}$$

4. STRESSES IN AN ANNULAR PLATE FIXED ALONG THE OUTER CONTOUR FOR GIVEN RADIAL AND TANGENTIAL DISPLACEMENTS OF THE CONTOUR OF THE CENTRAL HOLE

Consider a thin circular plate of unit thickness with a central circular hole into which is soldered or embedded a washer (or a rivet, if considering the technical applications of the solution of the problem, for instance, for joining two plates). In the corresponding technological operation the rivet will be deformed in such a way that plastic deformations will develop in it, as a result of which it is fixed in the sheet. The diameter of the cylindrical rivet increases under plastic deformation, which is one of the reasons for the generation of radial displacements of the inner contour in the annular plate. The instrument is sometimes turned in operations of expanding apertures when increasing the diameter; as a result, tangential displacements may appear along the inner contour. Precisely these considerations were the basis of the formulation of the problem considered in this section.

Hence, the first part of the boundary conditions can be reduced to the following. We will introduce a system of cylindrical polar coordinates (ρ, ϑ, x_3). We will denote the outer radius of the annular plate by R and the radius of its inner aperture by r. Then the boundary conditions of the problem can be written in the form

$$u_\rho + iu_\vartheta|_{\rho = r} = U_0 + iV_0, \quad u_\rho + iu_\vartheta|_{\rho = R} = 0 \tag{4.1}$$

We will also assume that the component of the stress tensor satisfies the condition

$$\sigma_{33}|_{\rho = R} = 0 \tag{4.2}$$

and that the component of the displacement vector satisfies the condition

$$u_3|_{\rho = r} = \varepsilon_0 x_3 \tag{4.3}$$

The last condition has meaning in cases when the washer is soldered into the plate or if there is friction between it and the plate. We draw attention to the possibility of relating the quantity U_0 to the displacement u_3, for instance, by using the conditions of incompressibility of the material of the rivet or any other condition.

We will also obtain a system of equations for the unknown coefficients of the expansions A, a_1, b_{-1}, c_1 (of these only A is a real quantity) as above, by using boundary conditions (4.1)–(4.3) and taking into account Eqs (1.1), (1.6)–(1.9) and using Eqs (1.14)–(1.16). The system obtained differs from system (2.17): the right-hand side of the first equation is now equal to $4\mu(U_0 + iV_0)$, and the right-hand side of the third equation is now equal to $4\mu\varepsilon_0$.

By splitting the unknown coefficients into real and imaginary parts, i.e. assuming that

$$a_1 = a_{11} + ia_{12}, \quad b_{-1} = b_{-11} + ib_{-12}, \quad c_1 = c_{11} + ic_{12} \tag{4.4}$$

we obtain two systems of equations for the real and imaginary parts of the coefficients of the expansions. The matrices of both systems correspond to the matrices (2.18). It is easy to see that both systems are compatible.

The solution of the system of equations for the imaginary parts gives

$$b_{-12} = \frac{2\mu V_0 r R^2}{R^2 - r^2}, \quad c_{12} = -\frac{4\mu V_0 r}{R^2 - r^2} \tag{4.5}$$

Note that the imaginary part of the coefficient a_1 from system (4.4) is not determined; however, it is not required for the determination of any mechanical quantity, which was already observed in the previous section.

The solution of the system of equations for the real parts of the coefficients of the expansions has the form

$$A = \frac{1}{\Delta}[2\nu U_0 r - (1-\nu)\varepsilon_0(R^2 - r^2)], \quad a_{11} = \frac{1}{2\Delta}(A_1 + A_2 + A_3)$$

$$b_{-11} = \frac{2rR}{\Delta}\left\{\nu U_0 R - [(1-\nu)\varepsilon_0 Rr + 4(1-\nu)^2 U_0 R]\ln\frac{R}{r}\right\} \tag{4.6}$$

$$c_{11} = \frac{2(2-\nu)}{\Delta}\left\{\varepsilon_0(R^2 - r^2) - 2[\varepsilon_0 r^2 + 4(1-\nu)U_0 r]\ln\frac{R}{r}\right\}$$

where

$$A_1 = [2(1-\nu)\varepsilon_0 R^2 - 2(2-\nu)(\varepsilon_0 r + 2U_0)r]\ln R$$

$$A_2 = 2[\varepsilon_0 r + 4(1-\nu)2U_0]r\ln r$$

$$A_3 = (3-2\nu)\varepsilon_0(R^2-r^2)-4\nu U_0 r$$

$$\Delta = \frac{1}{\mu}\left\{[2(1-\nu)^2R^2-(1-2\nu)(2-\nu)r^2]\ln\frac{R}{r}-\nu(R^2-r\right.$$

Omitting the simple mathematics, we reduce the expressions for components of the stress tensor to the form

$$\begin{aligned}
\sigma_{\rho\rho} &= \frac{c_{11}}{2}-2(1-2\nu)[A(\ln\rho+1)+a_{11}]-A+\frac{b_{-11}}{\rho^2}\\
\sigma_{\vartheta\vartheta} &= \frac{c_{11}}{2}-2(1-2\nu)[A(\ln\rho+1)+a_{11}]+A-\frac{b_{-11}}{\rho^2}\\
\sigma_{33} &= 4(2-\nu)[A(\ln\rho+1)+a_{11}]-c_{11}\\
\sigma_{\rho\vartheta} &= \frac{b_{-12}}{\rho^2},\quad \sigma_{\rho 3} = 4(1-\nu)A\frac{x_3}{\rho},\quad \sigma_{\vartheta 3} = 0
\end{aligned} \tag{4.7}$$

For the component ε_{33} of the strain tensor, directly required for verifying boundary condition (4.3), we have the expression

$$4\mu\varepsilon_{33} = 16(1-\nu)[A(\ln\rho+1)+a_{11}]-2c_{11}$$

The components of the displacement vector can also be easily determined

$$u_\rho = -2A\rho(\ln\rho+1)-4a_{11}\rho-2\frac{b_{-11}}{\rho}+c_{11}\rho,\quad u_\vartheta = 2\frac{b_{-12}}{\rho}+c_{12}\rho$$

We will now determine the limit values of the components of the stress tensor along the inner contour of the hole in the annular plate. To do this we will put $\rho = r \to 0$ in Eqs (4.7). As a result we obtain

$$\sigma_{\rho\rho} = -\mu\left[\varepsilon_0+2\frac{U_0}{r}\right],\quad \sigma_{\vartheta\vartheta} = \mu\left[\frac{\nu}{1-\nu}\varepsilon_0+2\frac{U_0}{r}\right],\quad \sigma_{33} = \frac{2-\nu}{1-\nu}\mu\varepsilon_0,\quad \sigma_{\rho\vartheta} = 2\mu\frac{V_0}{r},$$

$$\sigma_{\rho 3} = \mu x_3\frac{\varepsilon_0}{r\ln r}$$

We will now estimate the influence of the choice of the boundary conditions on the boundary of contact of the washer and the plate. We will assume that the free surface of the plate is not loaded; thus instead of condition (4.3) we will consider the boundary condition

$$\sigma_{33}|_{\rho=r} = 0$$

As above, the problem can be reduced to two systems of equations for the real and imaginary parts of the unknown coefficients of the expansions A, a_1, b_{-1}, c_1. The matrix of the system for real parts of the coefficients of the expansions (4.4) differs from the first matrix (2.18) only in that in its third row in the first and the second columns the factor $1-\nu$ is replaced by the factor $1-\nu/2$. The matrix for the imaginary parts corresponds to the second matrix of (2.18), by virtue of which the coefficients b_{-12}, c_{12} are also determined by relations (4.5).

The solution of the system of equations for the real parts of the coefficients of the expansions gives

$$A = 0,\quad a_{11} = -\frac{\mu U_0 r}{(1-\nu)(R^2-r^2)},\quad b_{-11} = -\frac{2\mu U_0R^2r}{(R^2-r^2)},\quad c_{11} = -\frac{4\mu(2-\nu)U_0 r}{(1-\nu)(R^2-r^2)}$$

The imaginary part of the coefficient a_1, as mentioned above, plays no part in the determination of the components of the stress tensor.

Using the expressions for these coefficients we obtain the following representations for the components of the stress tensor

$$\sigma_{\rho\rho} = -2\mu\frac{(1-\nu)R^2+(1+\nu)r^2}{(1-\nu)(R^2-r^2)}\frac{U_0 r}{\rho^2}$$

$$\sigma_{\vartheta\vartheta} = 2\mu\frac{(1-\nu)R^2-(1+\nu)r^2}{(1-\nu)(R^2-r^2)}\frac{U_0 r}{\rho^2}$$

$$\sigma_{\rho\vartheta} = 2\mu\frac{R^2}{(R^2-r^2)}\frac{V_0 r}{\rho^2}$$

The remaining components vanish.

For the components of the displacement vector we have

$$u_\rho = \frac{R^2-\rho^2}{R^2-r^2}\frac{U_0 r}{\rho}, \quad u_\vartheta = \frac{R^2-\rho^2}{R^2-r^2}\frac{V_0 r}{\rho}$$

And, finally, for the component ε_{33} of the strain tensor we obtain

$$\varepsilon_{33} = \frac{2\nu U_0 r}{(1-\nu)(R^2-r^2)}$$

To estimate the effect of concentrated loads we will put $\rho = r \to 0$. As a result we obtain

$$\sigma_{\rho\rho} = -2\mu\frac{U_0}{r}, \quad \sigma_{\vartheta\vartheta} = 2\mu\frac{U_0}{r}, \quad \sigma_{\rho\vartheta} = 2\mu\frac{V_0}{r}$$

Note that the limit values of the components $\sigma_{\rho\rho}$ and $\sigma_{\vartheta\vartheta}$ of the stress tensor correspond completely to the known expressions for these quantities for the given radial displacement [1].

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